A Relation between Coherent States and Generalized Bell States

Kazuyuki FUJII *†

Department of Mathematical Sciences

Yokohama City University

Yokohama 236-0027

JAPAN

Abstract

In the first half we show an interesting relation between coherent states and the Bell states in the case of spin 1/2, which was suggested by Fivel.

In the latter half we treat generalized coherent states and try to generalize this relation to get several generalized Bell states.

Our method is based on a geometry and our task may give a hint to open a deep relation between a coherence and an entanglement.

*E-mail address : fujii@math.yokohama-cu.ac.jp

†Home-page : http://fujii.sci.yokohama-cu.ac.jp

1 Introduction

The recent progress of quantum information theory including quantum computer, quantum cryptgraphy and quantum teleportation is marvelous enough. The coherence and entanglement play an essential role in quantum information theory. See the papers in [1] or [4].

In [2] Bell considered the so–called Bell states to test the EPR problem ("paradox") and proposed the famous inequality, see [3] or [4]. The Bell states are typical examples of the entanglement. Interestingly enough they have been used in the field of quantum teleportation. They are in the case of spin 1/2. Of course we can consider states with general spin j. We call them generalized Bell states.

On the other hand coherent states are fundamental tools in quantum optics and they are of course entangled. See [5]. Coherent states (generalized coherent states) are related with unitary representations of compact or non–compact Lie groups such as $U(n)$ or $U(n-1,1)$, see [6].

What is a relation between coherent states and Bell states or generalized Bell states ? We would like to construct a mathematical theory between them. In [7] Fivel defined the generalized Bell states as the integral of tensor product of generalized coherent state and its "twisted" one. We redefine Fivel's one to be more calculable and perform several integrals. Then we recover the Bell states and, moreover, get Bell states with general spin and more. In a certain sense the states of Fivel are overcomplete expression of Bell states or generalized Bell states.

By the way we are now developping Holonomic Quantum Computation, [14]–[21]. One of our aim of this study is to apply the idea of generalized Bell states to it. But we have a trouble. The Fivel's states are not defined for coherent states based on non–compact Lie group such as $U(n-1,1)$. This point is unsatisfactory to us. Therefore we need to extend our method more widely.

2 Review on General Theory

We make a review of [7] within our necessity. Let G be a compact linear Lie group (for example $G = U(n)$) and consider a coherent representation of G whose parameter space is a compact complex manifold $S = G/H$, where H is a subgroup of G. For example $G = U(n)$ and $H = U(k) \times U(n-k)$, then $S = U(n)/U(k) \times U(n-k) \cong G_k(\mathbf{C}^n)$, a complex Grassmann manifold. See in detail [6] or [10]. Let Z be a local coordinate and $|Z\rangle$ a generalized coherent state in some representation space V ($\cong \mathbf{C}^K$ for some big $K \in \mathbf{N}$). Then we have from the definition the measure $d\mu(Z, Z^\dagger)$ that satisfies the resolution of unity

$$\int_S d\mu(Z, Z^\dagger)|Z\rangle\langle Z| = \mathbf{1}_V \quad \text{and} \quad \int_S d\mu(Z, Z^\dagger) = \text{dim}V \ . \tag{1}$$

Next we define an anti-automorphism $\flat : S \longrightarrow S$. We call $Z \longrightarrow Z^\flat$ an anti-automorphism if and only if

$$\text{(i)} \quad Z \longrightarrow Z^\flat \text{ induces an automorphism of } S, \tag{2}$$

$$\text{(ii)} \quad \flat \text{ is an anti-map, namely} \quad \langle Z^\flat|W^\flat\rangle = \langle W|Z\rangle. \tag{3}$$

Now let us define the generalized Bell state [7] :

Definition (Fivel) The generalized Bell state is defined as

$$||B\rangle\rangle = \frac{1}{\sqrt{\text{dim}V}} \int_S d\mu(Z, Z^\dagger)|Z\rangle \otimes |Z^\flat\rangle. \tag{4}$$

Then we have

$$\begin{aligned}
\langle\langle B||B\rangle\rangle &= \frac{1}{\text{dim}V} \int_S \int_S d\mu(Z, Z^\dagger) d\mu(W, W^\dagger)(\langle Z| \otimes \langle Z^\flat|)(|W\rangle \otimes |W^\flat\rangle) \\
&= \frac{1}{\text{dim}V} \int_S \int_S d\mu(Z, Z^\dagger) d\mu(W, W^\dagger)\langle Z|W\rangle\langle Z^\flat|W^\flat\rangle \\
&= \frac{1}{\text{dim}V} \int_S \int_S d\mu(Z, Z^\dagger) d\mu(W, W^\dagger)\langle Z|W\rangle\langle W|Z\rangle \\
&= \frac{1}{\text{dim}V} \int_S d\mu(Z, Z^\dagger)\langle Z|Z\rangle = \frac{1}{\text{dim}V} \int_S d\mu(Z, Z^\dagger) = 1,
\end{aligned}$$

where we have used (1) and (3).

Therefore we can get several generalized Bell states as choosing several anti-automorphisms. In the next section we will show that these states just coincide with the famous Bell states [2] in the case of spin $\frac{1}{2}$:

$$\begin{aligned} &\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\ &\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle). \end{aligned} \tag{5}$$

Next we make a review of complex projective spaces, [11], [9] and [12]. For $N \in \mathbf{N}$ the complex projective space $\mathbf{C}P^N$ is defined as follows : For $\boldsymbol{\zeta}$, $\boldsymbol{\mu} \in \mathbf{C}^{N+1} - \{\mathbf{0}\}$ $\boldsymbol{\zeta}$ is equivalent to $\boldsymbol{\mu}$ ($\boldsymbol{\zeta} \sim \boldsymbol{\mu}$) if and only if $\boldsymbol{\zeta} = \lambda\, \boldsymbol{\mu}$ for some $\lambda \in \mathbf{C} - \{0\}$. We show its equivalence relation class as $[\boldsymbol{\zeta}]$ and set $\mathbf{C}P^N \equiv \mathbf{C}^{N+1} - \{\mathbf{0}\}/\sim$. When $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \cdots, \zeta_N)$ we write usually as $[\boldsymbol{\zeta}] = [\zeta_0 : \zeta_1 : \cdots : \zeta_N]$. Then it is well–known that $\mathbf{C}P^N$ has $N+1$ local charts, namely

$$\mathbf{C}P^N = \bigcup_{j=0}^{N} U_j \ , \quad U_j = \{[\zeta_0 : \cdots : \zeta_j : \cdots : \zeta_N] \mid \zeta_j \neq 0\}. \tag{6}$$

Since

$$(\zeta_0, \cdots, \zeta_j, \cdots, \zeta_N) = \zeta_j \left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, 1, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j} \right),$$

we have the local coordinate on U_j

$$\left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_N}{\zeta_j} \right). \tag{7}$$

But the above definition of $\mathbf{C}P^N$ is not handy, so we use the well–known expression by projections

$$\mathbf{C}P^N \cong G_1(\mathbf{C}^{N+1}) = \{P \in M(N+1;\mathbf{C}) \mid P^2 = P,\ P^\dagger = P \text{ and } \mathrm{tr}P = 1\} \tag{8}$$

and this correspondence

$$[\zeta_0 : \zeta_1 : \cdots : \zeta_N] \Longleftrightarrow \frac{1}{|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_N|^2} \begin{pmatrix} |\zeta_0|^2 & \zeta_0\bar{\zeta}_1 & \cdot & \cdot & \zeta_0\bar{\zeta}_N \\ \zeta_1\bar{\zeta}_0 & |\zeta_1|^2 & \cdot & \cdot & \zeta_1\bar{\zeta}_N \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \zeta_N\bar{\zeta}_0 & \zeta_N\bar{\zeta}_1 & \cdot & \cdot & |\zeta_N|^2 \end{pmatrix} \equiv P\ . \tag{9}$$

If we set

$$|\boldsymbol{\zeta}\rangle = \frac{1}{\sqrt{\sum_{j=0}^{N}|\zeta_j|^2}}\begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \cdot \\ \cdot \\ \zeta_N \end{pmatrix}, \tag{10}$$

then we can write the right hand side of (9) as

$$P = |\boldsymbol{\zeta}\rangle\langle\boldsymbol{\zeta}| \quad \text{and} \quad \langle\boldsymbol{\zeta}|\boldsymbol{\zeta}\rangle = 1. \tag{11}$$

For example on U_1

$$(z_1, z_2, \cdots, z_N) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}, \cdots, \frac{\zeta_N}{\zeta_0}\right),$$

we have

$$\begin{aligned} P(z_1, \cdots, z_N) &= \frac{1}{1+\sum_{j=1}^{N}|z_j|^2}\begin{pmatrix} 1 & \bar{z}_1 & \cdot & \cdot & \bar{z}_N \\ z_1 & |z_1|^2 & \cdot & \cdot & z_1\bar{z}_N \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ z_N & z_N\bar{z}_1 & \cdot & \cdot & |z_N|^2 \end{pmatrix} \\ &= |(z_1, z_2, \cdots, z_N)\rangle\langle(z_1, z_2, \cdots, z_N)|\ , \end{aligned} \tag{12}$$

where

$$|(z_1, z_2, \cdots, z_N)\rangle = \frac{1}{\sqrt{1+\sum_{j=1}^{N}|z_j|^2}}\begin{pmatrix} 1 \\ z_1 \\ \cdot \\ \cdot \\ z_N \end{pmatrix}. \tag{13}$$

For the latter use let us give a more detail description for the cases $N = 1$ and 2.

(a) $N = 1$:

$$P(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} = |z\rangle\langle z|,$$

$$\text{where } |z\rangle = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad z = \frac{\zeta_1}{\zeta_0}, \quad \text{on } U_1 \ , \tag{14}$$

$$P(w) = \frac{1}{|w|^2+1} \begin{pmatrix} |w|^2 & w \\ \bar{w} & 1 \end{pmatrix} = |w\rangle\langle w|,$$

$$\text{where } |w\rangle = \frac{1}{\sqrt{|w|^2+1}} \begin{pmatrix} w \\ 1 \end{pmatrix}, \quad w = \frac{\zeta_0}{\zeta_1}, \quad \text{on } U_2 \ . \tag{15}$$

(b) $N = 2$:

$$P(z_1, z_2) = \frac{1}{1+|z_1|^2+|z_2|^2} \begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & |z_1|^2 & z_1\bar{z}_2 \\ z_2 & z_2\bar{z}_1 & |z_2|^2 \end{pmatrix} = |(z_1, z_2)\rangle\langle(z_1, z_2)|,$$

$$\text{where} \quad |(z_1, z_2)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}} \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}, \quad (z_1, z_2) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}\right) \quad \text{on } U_1 \ , \tag{16}$$

$$P(w_1, w_2) = \frac{1}{|w_1|^2+1+|w_2|^2} \begin{pmatrix} |w_1|^2 & w_1 & w_1\bar{w}_2 \\ \bar{w}_1 & 1 & \bar{w}_2 \\ w_2\bar{w}_1 & w_2 & |w_2|^2 \end{pmatrix} = |(w_1, w_2)\rangle\langle(w_1, w_2)|,$$

$$\text{where} \quad |(w_1, w_2)\rangle = \frac{1}{\sqrt{|w_1|^2+1+|w_2|^2}} \begin{pmatrix} w_1 \\ 1 \\ w_2 \end{pmatrix}, \quad (w_1, w_2) = \left(\frac{\zeta_0}{\zeta_1}, \frac{\zeta_2}{\zeta_1}\right) \quad \text{on } U_2 \ , \tag{17}$$

$$P(v_1, v_2) = \frac{1}{|v_1|^2+|v_2|^2+1} \begin{pmatrix} |v_1|^2 & v_1\bar{v}_2 & v_1 \\ v_2\bar{v}_1 & |v_2|^2 & v_2 \\ \bar{v}_1 & \bar{v}_2 & 1 \end{pmatrix} = |(v_1, v_2)\rangle\langle(v_1, v_2)|,$$

$$\text{where} \quad |(v_1, v_2)\rangle = \frac{1}{\sqrt{|v_1|^2+|v_2|^2+1}} \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix}, \quad (v_1, v_2) = \left(\frac{\zeta_0}{\zeta_2}, \frac{\zeta_1}{\zeta_2}\right) \quad \text{on } U_3 \ . \tag{18}$$

3 Bell States Revisited

In this section we show that (4) coinsides with the Bell states (5) by choosing anti-automorphism $\flat$ suitably.

First let us recall the spin j–representation of Lie algebra $su(2)$ from [8]. This is a coherent representation of $su(2)$ based on complex manifold $\mathbf{C}P^1$ in our terminology. The algebra of $\{J_+, J_-, J_3\}$ reads

$$[J_3, J_+] = J_+,\ [J_3, J_-] = -J_-,\ [J_+, J_-] = 2J_3\ , \tag{19}$$

where $J_\pm = \frac{1}{2}(J_1 \pm iJ_2)$ and actions of $\{J_+, J_-, J_3\}$ on a representation space V ($\cong \mathbf{C}^{2j+1}$)

$$\begin{aligned}&J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle,\ J_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}|j,m-1\rangle,\\ &J_3|j,m\rangle = m|j,m\rangle,\end{aligned} \tag{20}$$

where $-j \le m \le j$. We note

$$\mathbf{1}_j = \sum_{m=-j}^{j} |j,m\rangle\langle j,m| \quad \text{and} \quad \langle j,m|j,n\rangle = \delta_{mn}. \tag{21}$$

Then the coherent state $|z\rangle$ ($z \in \mathbf{C} \subset \mathbf{C}P^1$) is defined as

$$|z\rangle = \frac{1}{\left(1+|z|^2\right)^j} \sum_{k=0}^{2j} \sqrt{{}_{2j}C_k}\, z^k |j, -j+k\rangle \tag{22}$$

and this satisfies the resolution of unity (1)

$$\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle\langle z| = \sum_{m=-j}^{j} |j,m\rangle\langle j,m| = \mathbf{1}_j \quad \text{and} \quad \int_{\mathbf{C}} d\mu(z,\bar{z}) = 2j+1\ , \tag{23}$$

where the measure $d\mu(z,\bar{z})$ is

$$d\mu(z,\bar{z}) = \frac{2j+1}{\pi} \frac{[d^2z]}{(1+|z|^2)^2}\ . \tag{24}$$

We note that this measure is invariant under the transform $z \longrightarrow 1/z$, so this one is defined on $\mathbf{C}P^1$ not $\mathbf{C}$. In the following we set for simplicity

$$|j, -j+k\rangle = |k\rangle \quad \text{for} \quad 0 \le k \le 2j\ . \tag{25}$$

For example $|\frac{1}{2},-\frac{1}{2}\rangle = |0\rangle$ and $|\frac{1}{2},\frac{1}{2}\rangle = |1\rangle$ in the case of spin $\frac{1}{2}$. In this case we consider the following four anti-automorphisms (2) and (3) :

$$(1)\ \ z^\flat = \bar{z} \qquad (2)\ \ z^\flat = -\bar{z} \qquad (3)\ \ z^\flat = \frac{1}{\bar{z}} \qquad (4)\ \ z^\flat = \frac{-1}{\bar{z}}\ . \tag{26}$$

Then it is easy to see from (14) and (15)

Lemma 1

$$(1)\ \ |z^\flat\rangle = |\bar{z}\rangle = \frac{1}{\sqrt{1+|z|^2}}(|0\rangle + \bar{z}|1\rangle), \tag{27}$$

$$(2)\ \ |z^\flat\rangle = |-\bar{z}\rangle = \frac{1}{\sqrt{1+|z|^2}}(|0\rangle - \bar{z}|1\rangle), \tag{28}$$

$$(3)\ \ |z^\flat\rangle = |1/\bar{z}\rangle = \frac{1}{\sqrt{1+|z|^2}}(\bar{z}|0\rangle + |1\rangle), \tag{29}$$

$$(4)\ \ |z^\flat\rangle = |-1/\bar{z}\rangle = \frac{1}{\sqrt{1+|z|^2}}(\bar{z}|0\rangle - |1\rangle). \tag{30}$$

Here we have identified $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. Then making use of elementary facts

$$\frac{2}{\pi}\int_{\mathbf{C}} \frac{[d^2z]}{(1+|z|^2)^2}\frac{1}{1+|z|^2} = \frac{2}{\pi}\int_{\mathbf{C}} \frac{[d^2z]}{(1+|z|^2)^2}\frac{|z|^2}{1+|z|^2} = 1,$$
$$\frac{2}{\pi}\int_{\mathbf{C}} \frac{[d^2z]}{(1+|z|^2)^2}\frac{z}{1+|z|^2} = \frac{2}{\pi}\int_{\mathbf{C}} \frac{[d^2z]}{(1+|z|^2)^2}\frac{\bar{z}}{1+|z|^2} = 0,$$

we have easily

Proposition 2

$$(1)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle \otimes |\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle), \tag{31}$$

$$(2)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle \otimes |-\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle), \tag{32}$$

$$(3)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle \otimes |1/\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle), \tag{33}$$

$$(4)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle \otimes |-1/\bar{z}\rangle = \frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle), \tag{34}$$

where $d\mu(z,\bar{z}) = \frac{2}{\pi}\frac{[d^2z]}{(1+|z|^2)^2}$. We note that our calculation is based on the following two matrices :

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\ . \tag{35}$$

We just recovered the Bell states (5) !! The author does not know whether this result has been known or not.

Since we consider the case of spin j, we write $|z\rangle$ as

$$|z\rangle_j = \frac{1}{\left(1+|z|^2\right)^j}\sum_{k=0}^{2j}\sqrt{{}_{2j}C_k}\,z^k|k\rangle \tag{36}$$

to emphasize the dependence of spin j. From Proposition 2 it is very natural to define Bell states with spin j as follows :

Definition 3 (Bell states with spin j)

$$(1)\quad ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle_j \otimes |\bar{z}\rangle_j, \tag{37}$$

$$(2)\quad ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle_j \otimes |-\bar{z}\rangle_j, \tag{38}$$

$$(3)\quad ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle_j \otimes |1/\bar{z}\rangle_j, \tag{39}$$

$$(4)\quad ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\int_{\mathbf{C}} d\mu(z,\bar{z})|z\rangle_j \otimes |-1/\bar{z}\rangle_j, \tag{40}$$

where $d\mu(z,\bar{z}) = \frac{2j+1}{\pi}\frac{[d^2z]}{(1+|z|^2)^2}$. Let us calculate $|\bar{z}\rangle_j$, $|-\bar{z}\rangle_j$, $|1/\bar{z}\rangle_j$ and $|-1/\bar{z}\rangle_j$. It is easy to see

Lemma 4

$$(1)\quad |\bar{z}\rangle_j = \frac{1}{\left(1+|z|^2\right)^j}\sum_{k=0}^{2j}\sqrt{{}_{2j}C_k}\,\bar{z}^k|k\rangle, \tag{41}$$

$$(2)\quad |-\bar{z}\rangle_j = \frac{1}{\left(1+|z|^2\right)^j}\sum_{k=0}^{2j}\sqrt{{}_{2j}C_k}(-1)^k\bar{z}^k|k\rangle, \tag{42}$$

$$(3)\quad |1/\bar{z}\rangle_j = \frac{1}{\left(1+|z|^2\right)^j}\sum_{k=0}^{2j}\sqrt{{}_{2j}C_k}\,\bar{z}^k|2j-k\rangle, \tag{43}$$

$$(4)\quad |-1/\bar{z}\rangle_j = \frac{1}{\left(1+|z|^2\right)^j}\sum_{k=0}^{2j}\sqrt{{}_{2j}C_k}(-1)^k\bar{z}^k|2j-k\rangle. \tag{44}$$

From this lemma and the elementary facts

$$\frac{2j+1}{\pi}\int_{\mathbf{C}}\frac{[d^2z]}{(1+|z|^2)^2}\frac{|z|^{2k}}{(1+|z|^2)^{2j}} = \frac{1}{{}_{2j}C_k} \quad \text{for} \quad 0 \le k \le 2j\ ,$$

we can give explicit forms to the Bell states with spin j in Definition 3 :

Proposition 5

$$(1)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\sum_{k=0}^{2j}|k\rangle\otimes|k\rangle, \tag{45}$$

$$(2)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\sum_{k=0}^{2j}(-1)^k|k\rangle\otimes|k\rangle, \tag{46}$$

$$(3)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\sum_{k=0}^{2j}|k\rangle\otimes|2j-k\rangle, \tag{47}$$

$$(4)\ \ ||B\rangle\rangle = \frac{1}{\sqrt{2j+1}}\sum_{k=0}^{2j}(-1)^k|k\rangle\otimes|2j-k\rangle. \tag{48}$$

A comment is in order. We list the above result once more for the case $j = 1$:

$$\begin{aligned}
&(1)\quad \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle + |2\rangle\otimes|2\rangle),\\
&(2)\quad \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle + |2\rangle\otimes|2\rangle),\\
&(3)\quad \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle + |1\rangle\otimes|1\rangle + |2\rangle\otimes|0\rangle),\\
&(4)\quad \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle - |1\rangle\otimes|1\rangle + |2\rangle\otimes|0\rangle).
\end{aligned}$$

It is easy to see that they are not linearly independent. Namely this case is very special (peculiar).

4 Generalized Bell States

In this section we generalize the result in the preceeding section, namely we treat the coherent states of $u(n+1)$ based on $\mathbf{C}P^n$ (see [9]) and calculate generalized Bell states (4) for several anti-automorphisms like (26). But to avoid complicated calculations we consider the case $n = 2$ and $Q = 1$ only, because it is easy to conjecture the corresponding result in general case from this special case.

Let $\{|0\rangle, |1\rangle, |2\rangle\}$ be a basis of the representation space V ($\cong \mathbf{C}^3$). Namely

$$\sum_{j=0}^{2} |j\rangle\langle j| = \mathbf{1}_{Q=1} \quad \text{and} \quad \langle i|j\rangle = \delta_{ij}.$$

A coherent state $|(z_1, z_2)\rangle$ for $(z_1, z_2) \in \mathbf{C}^2$ is defined as

$$|(z_1, z_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + z_1|1\rangle + z_2|2\rangle) \tag{49}$$

and the measure

$$d\mu(Z, Z^\dagger) = \frac{6}{\pi^2}\frac{[d^2 z_1][d^2 z_2]}{(1 + |z_1|^2 + |z_2|^2)^3}. \tag{50}$$

Then we have

$$\int_{\mathbf{C}^2} d\mu(Z, Z^\dagger)|(z_1, z_2)\rangle\langle(z_1, z_2)| = \sum_{j=0}^{2} |j\rangle\langle j| = \mathbf{1}_{Q=1} \quad \text{and} \quad \int_{\mathbf{C}^2} d\mu(Z, Z^\dagger) = 3\ . \tag{51}$$

Let ω be an element in $\mathbf{C}$ satisfying $\omega^3 = 1$. Then $1 + \omega + \omega^2 = 0$ and $\bar{\omega} = \omega^2$. Here we consider the following eight anti-automorphisms (2) and (3) :

$$\text{(a–1, 2, 3)} \quad (z_1, z_2)^\flat = (\bar{z}_1, \bar{z}_2),\ (z_1, z_2)^\flat = (\omega\bar{z}_1, \omega^2\bar{z}_2),\ (z_1, z_2)^\flat = (\omega^2\bar{z}_1, \omega\bar{z}_2), \tag{52}$$

$$\text{(b–1, 2, 3)} \quad (z_1, z_2)^\flat = (\frac{1}{\bar{z}_2}, \frac{\bar{z}_1}{\bar{z}_2}),\ (z_1, z_2)^\flat = (\omega\frac{1}{\bar{z}_2}, \omega^2\frac{\bar{z}_1}{\bar{z}_2}),\ (z_1, z_2)^\flat = (\omega^2\frac{1}{\bar{z}_2}, \omega\frac{\bar{z}_1}{\bar{z}_2}), \tag{53}$$

$$\text{(c–1, 2, 3)} \quad (z_1, z_2)^\flat = (\frac{\bar{z}_2}{\bar{z}_1}, \frac{1}{\bar{z}_1}),\ (z_1, z_2)^\flat = (\omega\frac{\bar{z}_2}{\bar{z}_1}, \omega^2\frac{1}{\bar{z}_1}),\ (z_1, z_2)^\flat = (\omega^2\frac{\bar{z}_2}{\bar{z}_1}, \omega\frac{1}{\bar{z}_1}). \tag{54}$$

A note is in order. We can of course choose another anti-automorphisms instead of the above ones.

Then it is easy to see from (16), (17) and (18)

Lemma 6

$$\text{(a–1)}\ |(z_1, z_2)^\flat\rangle = |(\bar{z}_1, \bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + \bar{z}_1|1\rangle + \bar{z}_2|2\rangle), \tag{55}$$

$$\text{(a–2)}\ |(z_1, z_2)^\flat\rangle = |(\omega\bar{z}_1, \omega^2\bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + \omega\bar{z}_1|1\rangle + \omega^2\bar{z}_2|2\rangle), \tag{56}$$

$$\text{(a–3)}\ |(z_1, z_2)^\flat\rangle = |(\omega^2\bar{z}_1, \omega\bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(|0\rangle + \omega^2\bar{z}_1|1\rangle + \omega\bar{z}_2|2\rangle), \tag{57}$$

$$\text{(b–1)}\ |(z_1, z_2)^\flat\rangle = |(1/\bar{z}_2, \bar{z}_1/\bar{z}_2)\rangle = \frac{1}{\sqrt{1 + |z_1|^2 + |z_2|^2}}(\bar{z}_2|0\rangle + |1\rangle + \bar{z}_1|2\rangle), \tag{58}$$

$$\text{(b–2)}\ |(z_1,z_2)^\flat\rangle = |(\omega/\bar{z}_2, \omega^2\bar{z}_1/\bar{z}_2)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}(\omega^2\bar{z}_2|0\rangle + |1\rangle + \omega\bar{z}_1|2\rangle), \quad (59)$$

$$\text{(b–3)}\ |(z_1,z_2)^\flat\rangle = |(\omega^2/\bar{z}_2, \omega\bar{z}_1/\bar{z}_2)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}(\omega\bar{z}_2|0\rangle + |1\rangle + \omega^2\bar{z}_1|2\rangle), \quad (60)$$

$$\text{(c–1)}\ |(z_1,z_2)^\flat\rangle = |(\bar{z}_2/\bar{z}_1, 1/\bar{z}_1)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}(\bar{z}_1|0\rangle + \bar{z}_2|1\rangle + |2\rangle), \quad (61)$$

$$\text{(c–2)}\ |(z_1,z_2)^\flat\rangle = |(\omega\bar{z}_2/\bar{z}_1, \omega^2/\bar{z}_1)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}(\omega\bar{z}_1|0\rangle + \omega^2\bar{z}_2|1\rangle + |2\rangle), \quad (62)$$

$$\text{(c–3)}\ |(z_1,z_2)^\flat\rangle = |(\omega^2\bar{z}_2/\bar{z}_1, \omega/\bar{z}_1)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}(\omega^2\bar{z}_1|0\rangle + \omega\bar{z}_2|1\rangle + |2\rangle). \quad (63)$$

From this lemma and the elementary facts

$$\frac{6}{\pi^2}\int_{\mathbf{C}^2}\frac{[d^2z_1][d^2z_2]}{(1+|z_1|^2+|z_2|^2)^3}\frac{1}{1+|z_1|^2+|z_2|^2} = \frac{6}{\pi^2}\int_{\mathbf{C}^2}\frac{[d^2z_1][d^2z_2]}{(1+|z_1|^2+|z_2|^2)^3}\frac{|z_1|^2}{1+|z_1|^2+|z_2|^2} =$$
$$\frac{6}{\pi^2}\int_{\mathbf{C}^2}\frac{[d^2z_1][d^2z_2]}{(1+|z_1|^2+|z_2|^2)^3}\frac{|z_2|^2}{1+|z_1|^2+|z_2|^2} = 1,$$

we have easily

Proposition 7 (generalized Bell states)

$$\text{(a–1)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle + |2\rangle\otimes|2\rangle), \quad (64)$$

$$\text{(a–2)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle + \omega|1\rangle\otimes|1\rangle + \omega^2|2\rangle\otimes|2\rangle), \quad (65)$$

$$\text{(a–3)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle + \omega^2|1\rangle\otimes|1\rangle + \omega|2\rangle\otimes|2\rangle), \quad (66)$$

$$\text{(b–1)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|2\rangle + |2\rangle\otimes|0\rangle), \quad (67)$$

$$\text{(b–2)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|1\rangle + \omega|1\rangle\otimes|2\rangle + \omega^2|2\rangle\otimes|0\rangle), \quad (68)$$

$$\text{(b–3)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|1\rangle + \omega^2|1\rangle\otimes|2\rangle + \omega|2\rangle\otimes|0\rangle), \quad (69)$$

$$\text{(c–1)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle + |1\rangle\otimes|0\rangle + |2\rangle\otimes|1\rangle), \quad (70)$$

$$\text{(c–2)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle + \omega|1\rangle\otimes|0\rangle + \omega^2|2\rangle\otimes|1\rangle), \quad (71)$$

$$\text{(c–3)}\quad ||B\rangle\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|2\rangle + \omega^2|1\rangle\otimes|0\rangle + \omega|2\rangle\otimes|1\rangle). \quad (72)$$

This is our main result. We note that our calculation is deeply based on the following two matrices :

$$A=\begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ 1 & & 0 \end{pmatrix}. \tag{73}$$

From Proposition 7 it is very easy to conjecture the explicit forms of generalized Bell states, [13].

5 Discussion

In this paper we calculated the generalized Bell states defined by Fivel, which are defined as the integral of tensor product of generalized coherent states based on $\mathbf{C}P^N$ and their "twisted" ones due to anti–automorphisms on $\mathbf{C}P^N$. The generalization to Grassmann manifolds $G_k(\mathbf{C}^{N+1})$ is under consideration, [13].

But unfortunately the generalized Bell states by Fivel are not defined for coherent states based on non–compact complex manifolds such as the Poincare disk or Siegel domains. We need a drastic change of his definition.

Acknowledgment. The author wishes to thank Yoshinori Machida for his warm hospitality at Numazu College of Technology.

References

[1] H-K. Lo, S. Popescu and T. Spiller (eds) : Introduction to Quantum Computation and Information, 1998, World Scientific.

[2] J. S. Bell : On the Einstein-Podolsky-Rosen paradox, Physics, 1(1964), 195.

[3] A. Peres : Quantum Theory : Concepts and Methods, 1995, Kluwer Academic Publishers.

[4] A. Hosoya : Lectures on Quantum Computation (in Japanese), 1999, Science Company (in Japan).

[5] L. Mandel and E. Wolf : Optical Coherence and Quantum Optics, 1995, Cambridge University Press.

[6] A. Perelomov : Generalized Coherent States and Their Applications, Springer–Verlag, 1986.

[7] D. I. Fivel : How a Quantum Theory Based on Generalized Coherent States Resolves the EPR and Measurement Problems, quant-ph/0104123.

[8] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Coherent states, path integral, and semiclassical approximation, J. Math. Phys., 36(1995), 3232.

[9] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Exactness in the Wentzel-Kramers-Brillouin approximation for some homogeneous spaces, J. Math. Phys., 36(1995), 4590.

[10] K. Fujii, T. Kashiwa, S. Sakoda :Coherent states over Grassmann manifolds and the WKB exactness in path integral, J. Math. Phys., 37(1996), 567.

[11] M. Nakahara : Geometry, Topology and Physics, IOP Publishing Ltd, 1990.

[12] K. Fujii : Introduction to Grassmann Manifold and Quantum Computation, quant-ph/0103011.

[13] K. Fujii : Geometry, Coherence and Entanglement (a tentative title), in progress.

[14] P. Zanardi and M. Rasetti : Holonomic Quantum Computation, Phys. Lett. A264(1999), 94, quant-ph/9904011.

[15] J. Pachos, P. Zanardi and M. Rasetti : Non-Abelian Berry connections for quantum computation, to appear in Phys. Rev. A, quant-ph/9907103.

[16] J. Pachos and S. Chountasis : Optical Holonomic Quantum Computer, quant-ph/9912093.

[17] J. Pachos and P. Zanardi : Quantum Holonomies for Quantum Computing, quant-ph/0007110.

[18] K. Fujii : Note on Coherent States and Adiabatic Connections, Curvatures, J. Math. Phys., 41(2000), 4406.

[19] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer, to appear in Rept. Math. Phys, quant-ph/0004102.

[20] K. Fujii : More on Optical Holonomic Quantum Computer, quant-ph/0005129.

[21] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer II, quant-ph/0101102.